AN ASYMPTOTIC FUNCTIONAL-INTEGRAL SOLUTION FOR THE SCHRÖDINGER EQUATION WITH POLYNOMIAL POTENTIAL

S. ALBEVERIO AND S. MAZZUCCHI

Abstract. A functional integral representation for the weak solution of the Schrödinger equation with a polynomially growing potential is proposed in terms of an analytically continued Wiener integral. The asymptotic expansion in powers of the coupling constant λ of the matrix elements of the Schrödinger group is studied and its Borel summability is proved.

Key words: Feynman path integrals, Schrödinger equation, analytic continuation of Wiener integrals, polynomial potential, asymptotic expansions.

AMS classification : 35C15, 35Q40, 35C20, 34M30, 28C20, 47D06, 35B60.

1. Introduction

The Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{1}$$

with a polynomial potential V of the form $V(x) = \lambda|x|^{2M}$ and the asymptotic behaviour of its solution in some limiting situation (for instance when $\lambda \to 0$ or $\hbar \to 0$, or $t \to 0$) is a largely studied topic [10, 28, 11, 9]. Particularly interesting is the study of a possible functional integral representation, in the spirit of Feynman path integrals.

During the last four decades, rigorous mathematical definitions of the heuristic Feynman path integrals have been given by means of different methods, and the properties of these rigorous integrals have been studied. let us mention here only three of them, namely the one using the analytic continuation of Wiener integrals [12, 13], the one provided by infinite dimensional oscillatory integrals [1, 23] and the one using white noise calculus [17] (see also the references given in [1, 23] to other approaches). The main problem which is common to all the existing approaches is the restriction on the class of potentials

V which can be handled. For most results one has to impose that V has at most quadratic growth at infinity. There are two exceptions to this restriction, the one of potentials which are Laplace transform of measures (like exponential potentials, see [1, 17] and references therein) and quartic potentials [4, 8, 22].

A difficulty in the study of equation (1) with a polynomial potential is the non regular behaviour of the solution. Indeed in [31] it has been shown that for superquadratic potentials the fundamental solution of (1) is nowhere of class C^1.

In [4, 8, 22] an infinite dimensional oscillatory integral representation for the weak solution of the Schrödinger equation, i.e. the matrix element of the Schrödinger group, has been presented and studied in the case where the potential has precisely a quartic growth at infinity.

The present paper generalizes partially these results to polynomial potentials with higher growth at infinity. For a dense set of vectors $\phi, \psi \in L^2(\mathbb{R}^d)$, we define an analytically continued Wiener integral $I_t^i(\phi, \psi)$ (equation (28)) which realizes rigorously the Feynman path integral representing the corresponding "matrix elements" of the Schrödinger group $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$ and prove that it solves the Schrödinger equation in a weak sense (theorem 8). The relation between the functional integral $I_t^i(\phi, \psi)$ and the matrix elements $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$ is investigated in details. In particular we prove that these quantities are asymptotically equivalent both as $t \to 0$ and as $\lambda \to 0$. The asymptotic expansion in powers of λ of $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$ is studied and its Borel summability is proved. This result allows one to recover the matrix elements of the Schrödinger group from the asymptotic expansion in powers of λ of the functional integral $I_t^i(\phi, \psi)$, which in this sense can be recognized as an asymptotic weak solution of the Schrödinger equation.

The paper is organized as follows. In section 2 the analyticity properties of the spectrum of the anharmonic oscillator Hamiltonian H and of the matrix elements $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$ of the Schrödinger group are studied. In section 3 the Borel summability of the asymptotic expansion in powers of the coupling constant λ (Dyson expansion) of the quantities $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$ is proved. Section 4 studies the definition and the properties of the functional integral $I_t^i(\phi, \psi)$, while section 5 investigates its relations with the matrix elements of the Schrödinger group and their asymptotic equivalence.

2. The Schrödinger equation with polynomial potential

Let us consider the quantum anharmonic oscillator Hamiltonian with polynomial potential on $L^2(\mathbb{R}^d)$, that is the operator defined on the vectors $\phi \in C_0^\infty(\mathbb{R}^d)$ by

$$H\phi(x) = -\frac{\hbar^2}{2}\Delta\psi(x) + \lambda V_{2M}(x)\psi(x), \qquad x \in \mathbb{R}^d, \tag{2}$$

where $\lambda \in \mathbb{R}$ is a real positive "coupling" constant, V_{2M} is a positive homogeneous $2M$-order polynomial, and $\hbar$ is the reduced Planck's constant (the mass of the particle is set equal to 1 for simplicity). In the following, in order to simplify some notations, we shall put $V_{2M}(x) := |x|^{2M}$, but all our results are also valid in the more general case as they depend only on the positivity and the homogeneity properties of the potential V_{2M}.

H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ (see [26], theorem X.28). Its closure, denoted again by H, has the following domain:

$$\begin{aligned} D(H) &= D(\Delta) \cap D(|x|^{2M}) \\ &= \{\phi \in L^2(\mathbb{R}^d) : \int |x|^{4M}|\phi(x)|^2 dx < \infty, \int |p|^4|\hat{\phi}(p)|^2 dp < \infty\}, \end{aligned}$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ. It is well known that H is a positive operator with a pure point spectrum $\{E_n\} \subset \mathbb{R}^+$. Therefore $-H$ generates an analytic contraction semigroup, $P(z) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $P(z) = e^{-Hz}$, with z being a complex parameter with positive real part.

In the case where z is purely imaginary of the form $z = \frac{i}{\hbar}t$, $t \in \mathbb{R}$, one obtains a one parameter group of unitary operators $U(t) := e^{-\frac{i}{\hbar}Ht}$, i.e. the Schrödinger group. Given a vector $\psi_0 \in L^2(\mathbb{R}^d)$, the vector $\psi(t) := U(t)\psi_0$ belongs to $D(H)$ and it satisfies the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(t) = H\psi(t). \tag{3}$$

The particular form of the potential allows one to prove a scaling property for the eigenvalues $\{E_n\}$ of the operator H as well as their analyticity as a function of the coupling constant λ on a suitable region of a Riemann surface. The present lemma is taken from [28], which presents a detailed study of this problem, also in more general cases.

Lemma 1. *Let $E_n(\lambda)$ denote the $n-th$ eigenvalue of the Hamiltonian* (2)*. Then for $\lambda, \alpha > 0$ one has*

$$E_n(\lambda) = \alpha^{-1}E_n(\lambda\alpha^{M+1}) = . \tag{4}$$

In particular

$$E_n(\lambda) = \lambda^{\frac{1}{M+1}}E_n(1) \tag{5}$$

Proof: Let us consider, for any $\alpha \in \mathbb{R}^+$, the unitary operator $V(\alpha) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given on vectors $\phi \in S(\mathbb{R}^d)$ by:

$$V(\alpha)\phi(x) = \alpha^{1/4}\phi(\alpha^{1/2}x), \qquad x \in \mathbb{R}^d. \tag{6}$$

It it simple to verify that $V(\alpha)$ leaves $D(-\Delta)$ and $D(|x|^{2M})$ invariant and

$$V(\alpha)x^{2M}V(\alpha)^{-1} = \alpha^M x^{2M},$$
$$V(\alpha)\Delta V(\alpha)^{-1} = \alpha^{-1}\Delta.$$

It follows that

$$V(\alpha)HV(\alpha)^{-1} = \alpha^{-1}\Big(-\frac{\hbar^2}{2}\Delta + \alpha^{M+1}\lambda x^{2M}\Big).$$

In particular, by taking $\alpha = \lambda^{-1/(M+1)}$ one has

$$V(\lambda^{-1/(M+1)})HV(\lambda^{-1/(M+1)})^{-1} = \lambda^{1/(M+1)}\Big(-\frac{\hbar^2}{2}\Delta + x^{2M}\Big). \tag{7}$$

As for any $\alpha \in \mathbb{R}^+$, the operator $V(\alpha)HV(\alpha)^{-1}$ has the same spectrum of H, from equation (7) one easily deduces equation (5) □

Remark 1. *By analytic continuation, relation* (5) *allows to extends* $E_n(\lambda)$ *to all complex values of* λ *belonging to a Riemann surface. In particular, the function* E_n *is many-sheeted and has a* $(M+1)$*-st order branch point at* $\lambda = 0$.

Let us consider now the matrix elements of the evolution operator $U(t) = e^{-\frac{i}{\hbar}Ht}$, i.e. the inner products $\langle \phi, U(t)\psi\rangle$, with $\phi, \psi \in L^2(\mathbb{R}^d)$. Let us consider also the function $f : \mathbb{R}^+ \to \mathbb{C}$ of the coupling constant λ (present in H, hence in $U(t)$) defined by

$$f(\lambda) := \langle \phi, U(t)\psi\rangle, \qquad \lambda \in \mathbb{R}, \lambda > 0. \tag{8}$$

Let us denote by D_{θ_1,θ_2} the sector of of the Riemann surface $\tilde{\mathbb{C}}$ of the logarithm defined by:

$$D_{\theta_1,\theta_2} := \{z \in \mathbb{C}, z = \rho e^{i\phi} \ : \ \rho > 0, \phi \in (\theta_1, \theta_2)\}.$$

Let us consider the dense subset of $L^2(\mathbb{R}^d)$ made of finite linear combinations of vectors of the form

$$\phi(x) = P(x)e^{-\sigma^2|x|^2}, \qquad x \in \mathbb{R}^d \tag{9}$$

with P being any polynomial with complex coefficients and $\sigma^2 \in \mathbb{C}$ a complex constant with positive real part (that these vectors are dense in $L^2(\mathbb{R}^d)$ follows from the known fact that the finite linear combinations of Hermite functions are dense in the same space).

Theorem 1. *Let* $\phi, \psi \in S(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$*, such that the functions* $\bar{\phi}, \psi \in S(\mathbb{R}^d)$ *are of the form* (9)*, with* $\sigma^2 \in \mathbb{C}$*,* $\sigma^2 = |\sigma^2|e^{i\delta}$*,* $\delta \in \mathbb{R}$ *such that there exist an* $\epsilon > 0$ *with*

$$\cos(\delta + \alpha) > \epsilon, \qquad \forall \alpha \in \Big(0, \frac{M-1}{M+1}\pi\Big), \tag{10}$$

(This is the case for instance if $\delta = -\pi\frac{M-1}{2(M+1)}$*). Then the function* $f : \mathbb{R}^+ \to \mathbb{C}$ *defined by* (8) *can be extended to an analytic function on the sector* $D_{-(M-1)\pi,0}$ *of the Riemann surface* $\tilde{\mathbb{C}}$ *of the logarithm.*

Proof: Let us denote by $\{E_n(\lambda)\}$, resp. $\{e_n(\lambda)\}$ the eigenvalues and resp. the eigenvectors of the Hamiltonian operator (2). Under the given assumptions on ϕ, ψ and by equation (5), for $\lambda \in \mathbb{R}^+$ the function f is given by

$$f(\lambda) = \sum a_n(\lambda) b_n(\lambda) e^{-\frac{i}{\hbar}E_n(\lambda)t} = \sum a_n(\lambda) b_n(\lambda) e^{-\frac{i}{\hbar}\lambda^{\frac{1}{M+1}}E_n(1)t}$$

with

$$a_n(\lambda) = \langle \phi, e_n(\lambda)\rangle, \qquad b_n(\lambda) = \langle e_n(\lambda), \psi\rangle.$$

On the other hand each coefficient a_n, b_n can be extended to an analytic function of the variable λ on $D_{-(M-1)\pi,0}$. Indeed one has (for $\lambda > 0$) that $e_n(\lambda) = V(\lambda^{-1/(M+1)})^{-1}e_n(1)$, where $V(-\lambda^{1/(M+1)})$ is the operator defined by (6). Without loss of generality, we can consider as an instance a vector ψ of the form $\psi(x) = x^k e^{-\sigma^2 x^2}$. In this case one has:

$$\begin{aligned} b_n(\lambda) &= \langle V(\lambda^{-1/(M+1)})^{-1}e_n(1), \psi\rangle = \langle e_n(1), V(\lambda^{-1/(M+1)})\psi\rangle \quad (11)\\ &= \lambda^{-\frac{1}{4(M+1)}} \int e_n(1)(x) \lambda^{-\frac{k}{2(M+1)}} |x|^k e^{-\sigma^2 \lambda^{-\frac{1}{(M+1)}}|x|^2} \quad (12)\end{aligned}$$

and the coefficient $b_n(\lambda)$ can be interpreted in terms of the inner product between the vector $e_n(1)$ and the function

$$x \mapsto \psi_\lambda(x) := \lambda^{-\frac{1}{4(M+1)}} \lambda^{-\frac{k}{2(M+1)}} |x|^k e^{-\sigma^2 \lambda^{-\frac{1}{(M+1)}}|x|^2}. \tag{13}$$

For $\lambda \in D_{-(M-1)\pi,0}$ and for σ^2 satisfying the assumptions of the theorem, it is simple to verify that the function (13) belongs to $L^2(\mathbb{R}^d)$ and its L^2-norm is uniformly bounded in $D_{-(M-1)\pi,0}$:

$$\|\psi_\lambda\| < \frac{C_k}{\epsilon^{k+1/2}},$$

where the constant C_k depends on k, while ϵ is the parameter appearing in condition (10). An analogous reasoning holds also for the coefficients $a_n(\lambda)$.

On the other hand, for $\lambda \in D_{-(M-1)\pi,0}$, one has

$$|e^{-\frac{i}{\hbar}\lambda^{\frac{1}{M+1}}E_n(1)t}| \leq 1.$$

The function f is then given by the following limit of a sequence of analytic functions on $D_{-(M-1)\pi,0}$, uniformly bounded on it:

$$f(\lambda) = \lim_{N\to\infty} \sum_n^N a_n(\lambda) b_n(\lambda) e^{-\frac{i}{\hbar}\lambda^{\frac{1}{M+1}}E_n(1)t},$$

and by Vitali's theorem, the limit defines an analytic function on $D_{-(M-1)\pi,0}$. □

In the case where the potential is not homogeneous, in particular if we add to it a a second degree term, i.e. a term of the harmonic oscillator type, the proof of theorem 1 does not work. In particular, in the case where $H = H_0 + \lambda V_{2M}$, with $H_0 = -\frac{\hbar^2}{2}\Delta + \frac{\Omega^2}{2}x^2$, Ω^2 a $d \times d$ symmetric positive matrix and $V_{2M}(x) = |x|^{2M}$, an analogous result can only be obtained by further restricting the region of analyticity, as exposed in the following:

Theorem 2. *For $H = H_0 + \lambda V_{2M}$, with $H_0 = -\frac{\hbar^2}{2}\Delta + \frac{\Omega^2}{2}x^2$ and $V_{2M}(x) = |x|^{2M}$, the function $f : \mathbb{R}^+ \to \mathbb{C}$ given by $f(\lambda) := \langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$ can be extended to an analytic function of the complex variable λ in the sector $D_{-\pi,0}$.*

Proof:

By Trotter's product formula, for $\lambda \in \mathbb{R}^+$ one has for any $\phi, \psi \in L^2(\mathbb{R}^d)$:

$$f(\lambda) = \lim_{n\to\infty} \langle \phi, (e^{-\frac{it}{n\hbar}H_0} e^{-\frac{it}{n\hbar}\lambda V_{2M}})^n \psi\rangle.$$

The positive multiplication operator V_{2M} generates an analytic contraction semigroup, and for any $n \in \mathbb{N}$, the function $f_n : D_{-\pi,0} \to \mathbb{C}$ defined by

$$f_n(\lambda) := \langle \phi, (e^{-\frac{it}{n\hbar}H_0} e^{-\frac{it}{n\hbar}\lambda V_{2M}})^n \psi\rangle$$

is analytic on $D_{-\pi,0}$ and satisfies the bound $|f_n(\lambda)| \leq \|\phi\|\,\|\psi\|$. By Vitali's theorem the functions f_n converge to an analytic function f on $D_{-(M-1)\pi,0}$. □

3. Borel summability of the Dyson expansion of the Schrödinger group in powers of the coupling constant

Let us consider now the asymptotic expansion of the function f when $\lambda \to 0$. The present section is devoted to the proof of its Borel summability. We recall that an asymptotic expansion $\sum a_n z^n$ of a function

$f(z)$ as $z \to 0$ in an appropriate region of the complex plane is said to be *Borel summable* [16, 30] if the following procedure is possible:

(1) $B(t) = \sum a_n t^n/n!$ converges in some circle $|t| < r$;
(2) $B(t)$ has an analytic continuation to a neighborhood of the positive real axis;
(3) f can be computed in terms of the Borel-Laplace transform of $B(t)$, i.e. $f(z) = \frac{1}{z}\int_0^\infty e^{-t/z}B(t)dt$.

In this case the asymptotic expansion $\sum a_n z^n$ allows to construct the function f without any ambiguity and to characterize it uniquely. One of the main tools for the proof of Borel summability is Watson's theorem (and its improved version, i.e. Nevanlinna's theorem). We give here for later use a particular form of it [30, 27]:

Theorem 3. *Let $f(z)$ be an analytic function in a sectorial region of the Riemann surface $\tilde{\mathbb{C}}$ of the logarithm*

$$\{z \in \tilde{\mathbb{C}} | 0 < |z| < R, |arg(z)| < \frac{1}{2}k\pi\}$$

and satisfying there an estimate of the form

$$|f(z) - \sum^{N-1} a_n z^n| \leq AC^N |z|^N (kN)!$$

uniformly in N and z in the sector.
Then the asymptotic series $\sum a_n z^n$ is Borel summable to the function f.

Let us consider the function f of the complex variable λ given by $f(\lambda) = \langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$, where H is given by (2), and describing the matrix elements of the Schrödinger group. Its asymptotic expansion as $\lambda \to 0$ can be obtained in terms of the Dyson expansion of the evolution operator $U_\lambda(t) = e^{-\frac{i}{\hbar}Ht}$. In the following we shall denote by H_0 the Hamiltonian operator H in the case where $\lambda = 0$.

Lemma 2. *If $\lambda \in \mathbb{R}^+$, $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in L^2(\mathbb{R}^d)$, the function $f(\lambda) = \langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$ describing the Schrödinger group has the following asymptotic expansion:*

$$f(\lambda) = \sum_n^{N-1} a_n \lambda^n + R_N(\lambda) \tag{14}$$

with

$$a_n = \frac{1}{n!}\big(-\frac{i}{\hbar}\big)^n \int_0^t \dots \int_0^t \langle V_{2M}(-s_1)\dots V_{2M}(-s_n)U_0(-t)\phi, \psi\rangle ds_1 \dots ds_n$$

and

$$R_N(\lambda) = \frac{\lambda^N}{N!}\big(-\frac{i}{\hbar}\big)^N \int_0^t \dots \int_0^t$$
$$\langle V(-s_N)\dots V(-s_1)U_0(-t)\phi, U_0(-s_N)U_\lambda(s_N)\psi\rangle ds_1\dots ds_N \quad (15)$$

where $V(s_i) := U_0(-s_i)V_{2M}U_0(s_i)$, $U_0(t) = e^{-\frac{it}{\hbar}H_0}$.

Proof: The asymptotic expansion can be obtained by means of Dyson's expansion. Let us set $U_\lambda(t) = e^{-\frac{it}{\hbar}H}$ and set $U_0(t) = e^{-\frac{it}{\hbar}H_0}$. Let $\lambda \in \mathbb{R}^+$. Given a vector $\psi \in D(H_\lambda) = D(H_0)\cap D(V_{2M})$ one can easily prove that the vector $U_\lambda(t)\psi$ satisfies the following integral equation

$$U_\lambda(t)\psi = U_0(t)\psi - \frac{i\lambda}{\hbar}\int_0^t U_0(t-s)V_{2M}U_\lambda(s)\psi ds$$

so that for any $\phi \in L^2(\mathbb{R}^d)$, one has

$$f(\lambda) = \langle\phi, U_\lambda(t)\psi\rangle = \langle\phi, U_0(t)\psi\rangle - \frac{i\lambda}{\hbar}\int_0^t \langle\phi, U_0(t-s)V_{2M}U_\lambda(s)\psi\rangle ds$$

By choosing $\phi \in \mathcal{S}(\mathbb{R}^d)$, we have $\phi \in D[(U_0(s)V_{2M})^n]$ for any $s \in [0,t]$ and $n \in \mathbb{N}$, and one can easily prove that for any $N \in \mathbb{N}$ the following holds

$$\langle\phi, U_\lambda(t)\psi\rangle = \sum_n^{N-1} a_n\lambda^n + R_N(\lambda) \quad (16)$$

where

$$a_n = \frac{1}{n!}\big(-\frac{i}{\hbar}\big)^n \int_0^t \dots \int_0^t \langle V_{2M}(-s_1)\dots V_{2M}(-s_n)U_0(-t)\phi, \psi\rangle ds_1\dots ds_n$$

and

$$R_N(\lambda) = \frac{\lambda^N}{N!}\big(-\frac{i}{\hbar}\big)^N \int_0^t \dots \int_0^t$$
$$\langle V(-s_N)\dots V(-s_1)U_0(-t)\phi, U_0(-s_N)U_\lambda(s_N)\psi\rangle ds_1\dots ds_N \quad (17)$$

with $V(s_i) := U_0(-s_i)V_{2M}U_0(s_i)$.
Both sides of (16) are continuous functionals of the vector $\psi \in D(H)$ and can be extended to $\psi \in L^2(\mathbb{R}^d)$. □

Lemma 3. *Let ϕ, ψ satisfy the assumptions of theorem 1. Then the asymptotic expansion* (14) *holds in the whole analyticity region* $D_{-(M-1)\pi,0}$.

Proof: Under the stated assumptions, both sides of (14) are analytic in $D_{-(M-1)\pi,0}$ and coincides on $\mathbb{R}^+$. By the uniqueness of analytic continuation they coincide in the whole sector $D_{-(M-1)\pi,0}$. $\square$

The following Theorem gives the Borel summability property of the asymptotic expansion (14) for $\psi \in L^2(\mathbb{R}^d)$ and ϕ belonging to a dense set of vectors in $L^2(\mathbb{R}^d)$.

Theorem 4. *Let ϕ, ψ satisfy the assumptions of theorem 1. Then the asymptotic expansion* (14) *of the function f describing the Schrödinger group is Borel summable.*

Proof:

By theorem 1 the function f is analytic on a sector of amplitude $\pi(M-1)$ and admits there an asymptotic expansion of the form (14). By exploiting the particular form of the vector ϕ, by a direct computation it is possible to verify that the remainder R_N satisfies uniformly in $D_{-(M-1)\pi,0}$ the bound

$$|R_N(\lambda)| \leq AC^N|\lambda|^N\Gamma(N(M-1)),$$

where A, C are constants depending on ϕ, ψ.

By Nevanlinna's theorem [25, 30], the asymptotic expansion (14) is Borel summable. $\square$

Remark 2. *In the case $V(x) = |x|^4$ the Borel summability of the asymptotic expansion* (14) *can be proved also in the case where H_0 is the harmonic oscillator Hamiltonian, by using the result of theorem 2.*

4. On the functional integral representation of the weak solution of the Schrödinger equation

Let us consider the heuristic Feynman path integral representation for the matrix elements of the Schödinger group generated by the Hamiltonian (2):

$$\begin{aligned} \langle\phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle &=\text{''} \int_{\mathbb{R}^d} dx\bar{\phi}(x) \int_{\gamma(t)=x} e^{\frac{i}{\hbar}S_t(\gamma)}\psi(0,\gamma(0))d\gamma\text{''} \\ =\text{''} \int_{\mathbb{R}^d} dx\bar{\phi}(x) &\int_{\gamma(t)=0} e^{\frac{i}{2\hbar}\int_0^t|\dot{\gamma}(s)|^2ds}e^{-\frac{i\lambda}{\hbar}\int_0^t|\gamma(s)+x|^{2M}ds}\psi(0,\gamma(0)+x)d\gamma\text{''} \end{aligned} \tag{18}$$

where $\phi, \psi \in L^2(\mathbb{R}^d)$. The aim of the present section is to provide a rigorous mathematical definition of the right hand side of equation

(18) in terms of a well defined functional integral, by means of an analytically continued Wiener integral.

Let us consider first of all the heat equation

$$\hbar\frac{\partial}{\partial t}\psi = -zH\psi \tag{19}$$

where z is a positive real parameter. It is well known [26] that the self-adjoint operator H defined by closure from its restriction to the Schwartz space of test functions $S(\mathbb{R}^d)$ by (2) is the generator of an analytic semigroup. In particular given two vectors $\phi,\psi\in L^2(\mathbb{R}^d)$, the inner product $\langle\phi, e^{-\frac{z}{\hbar}Ht}\psi\rangle$ is given by the Feynman-Kac formula [29]:

$$\begin{aligned}\langle\phi, e^{-\frac{z}{\hbar}Ht}\psi\rangle &= \int_{\mathbb{R}^d}\bar\phi(x)\int_{C_t} e^{-\frac{z\lambda}{\hbar}\int_0^t|\sqrt{\hbar z}\omega(s)+x|^{2M}ds}\psi_0(\sqrt{\hbar z}\omega(t)+x)W(d\omega)dx\\ &= z^{d/2}\int_{\mathbb{R}^d}\bar\phi(\sqrt{z}x)\int_{C_t} e^{-\frac{z^{M+1}\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega(s)+x|^{2M}ds}\psi_0(\sqrt{\hbar z}\omega(t)+\sqrt{z}x)W(d\omega)dx\end{aligned} \tag{20}$$

By the analyticity property of the semigroup generated by H, one can easily deduce the following result:

Theorem 5. *The left hand side of* (20)*, namely the matrix element* $\langle\phi, e^{-\frac{z}{\hbar}Ht}\psi\rangle$*, extends, for any* $\phi,\psi\in L^2(\mathbb{R}^d)$*, to an analytic function of the complex variable* z*, holomorphic in* $D_{-\pi/2,\pi/2}$ *and continuous on the boundary* $\bar D_{-\pi/2,\pi/2}$*. In particular for* $z=i$ *one obtains the matrix elements of the Schrödinger group* $\langle\phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$*.*

By imposing suitable analyticity conditions on the vectors ϕ,ψ in (20) one obtains the following:

Theorem 6. *Let* $\bar\phi,\psi\in L^2(\mathbb{R}^d)$ *satisfying the following conditions:*

(1) *For any* $x\in\mathbb{R}^d$) *the functions*

$$z\mapsto\bar\phi(\sqrt{z}x),\quad z\mapsto\psi(\sqrt{z}x),\qquad z\in\bar D_{-\frac{\pi}{2(M+1)},\frac{\pi}{2(M+1)}}$$

are continuous on $\bar D_{-\frac{\pi}{2(M+1)},\frac{\pi}{2(M+1)}}$ *and holomorphic on* $D_{-\frac{\pi}{2(M+1)},\frac{\pi}{2(M+1)}}$

(2) *for any* $z\in\bar D_{-\frac{\pi}{2(M+1)},\frac{\pi}{2(M+1)}}$*, the functions*

$$x\mapsto\bar\phi(\sqrt{z}x),\quad x\mapsto\psi(\sqrt{z}x),\qquad x\in\mathbb{R}^d$$

belong to $L^2(\mathbb{R}^d)$*.*

Then the right hand side of (20)*, namely the integral*

$$I_t^z(\phi,\psi) := z^{d/2}\int_{\mathbb{R}^d}\bar\phi(\sqrt{z}x)\int_{C_t} e^{-\frac{z^{M+1}\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega(s)+x|^{2M}ds}\psi_0(\sqrt{\hbar z}\omega(t)+\sqrt{z}x)W(d\omega) \tag{21}$$

extends, for any $\phi, \psi \in L^2(\mathbb{R}^d)$, *to an analytic function of the complex variable* z, *holomorphic in* $D_{-\frac{\pi}{2(M+1)},\frac{\pi}{2(M+1)}}$ *and continuous on the boundary* $\bar{D}_{-\frac{\pi}{2(M+1)},\frac{\pi}{2(M+1)}}$.

Proof: By the stated assumptions, for any $z \in \bar{D}_{-\frac{\pi}{2(M+1)},\frac{\pi}{2(M+1)}}$, the integral (21) is well defined, and

$$\begin{aligned} |I_t^z(\phi,\psi)| &\leq z^{d/2}\int_{\mathbb{R}^d}|\bar{\phi}(\sqrt{z}x)|\int_{C_t}|\psi(\sqrt{\hbar z}\omega(t)+\sqrt{z}x)|W(d\omega) \\ &= |z|^{d/2}\langle\phi_z, e^{-\frac{1}{\hbar}H_0 t}\psi_z\rangle \leq \|\phi_z\|\|\psi_z\| \end{aligned} \tag{22}$$

where $\phi_z, \psi_z \in L^2(\mathbb{R}^d)$ are defined resp. by $\phi_z(x) := |\bar{\phi}(\sqrt{z}x)|$,$\psi_z(x) := |\psi(\sqrt{z}x)|$.

The analyticity of the function $z \mapsto I_t^z(\phi,\psi)$ on $D_{-\frac{\pi}{2(M+1)},\frac{\pi}{2(M+1)}}$ follows by Fubini's and by Morera's theorems. The continuity on $\bar{D}_{-\frac{\pi}{2(M+1)},\frac{\pi}{2(M+1)}}$ follows by the dominated convergence theorem. □

From the previous theorem one can easily deduce the following

Corollary 1. *Let* $\bar{\phi}, \psi \in L^2(\mathbb{R}^d)$ *satisfy the following conditions:*

(1) *For any* $x \in \mathbb{R}^d$) *the functions*

$$z \mapsto \bar{\phi}(\sqrt{z}x), \quad z \mapsto \psi(\sqrt{z}x), \qquad z \in \bar{D}_{0,\frac{\pi}{2}}$$

are continuous on $\bar{D}_{0,\frac{\pi}{2}}$ *and holomorphic on* $D_{0,\frac{\pi}{2}}$

(2) *for any* $z \in \bar{D}_{0,\frac{\pi}{2}}$, *the functions*

$$x \mapsto \bar{\phi}(\sqrt{z}x), \quad x \mapsto \psi(\sqrt{z}x), \qquad x \in \mathbb{R}^d$$

belong to $L^2(\mathbb{R}^d)$

Then

$$\langle\phi, e^{-\frac{i}{\hbar}H_0 t}\psi\rangle = e^{i\frac{\pi}{4}d}\int_{\mathbb{R}^d}\bar{\phi}(e^{i\frac{\pi}{4}}x)\int_{C_t}\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t)+e^{i\frac{\pi}{4}}x)W(d\omega)dx, \tag{23}$$

where H_0 *is the free Hamiltonian given on* $\psi \in C_0^2(\mathbb{R}^d)$ *by*

$$H_0\psi(x) = -\frac{\hbar^2}{2}\Delta\psi(x).$$

Proof: Let us consider the function $f_1 : \bar{D}_{-\pi/2,\pi/2} \to \mathbb{C}$ given by $f_1(z) = \langle\phi, e^{-\frac{z}{\hbar}H_0}\psi\rangle$. By theorem 5, f_1 is analytic on $D_{-\pi/2,\pi/2}$ and continuous on $\bar{D}_{-\pi/2,\pi/2}$.

Let $f_2 : \bar{D}_{0,\frac{\pi}{2}} \to \mathbb{C}$ be defined by

$$f_2(z) = z^{d/2}\int_{\mathbb{R}^d}\bar{\phi}(\sqrt{z}x)\int_{C_t}\psi(\sqrt{\hbar}\sqrt{z}\omega(t)+\sqrt{z}x)W(d\omega)dx.$$

By theorem 6, f_2 is analytic on $D_{0,\pi/2}$ and continuous on the closure $\bar{D}_{0,\pi/2}$ of $D_{0,\pi/2}$.
By the Feynman-Kac formula (20), the functions f_1 and f_2 coincide on $\mathbb{R}^+$. By the uniqueness of analytic continuation they coincide on the whole domain. In particular, by the continuity on the boundary, one has $f_1(i) = f_2(i)$, i.e.

$$\langle \phi, e^{-\frac{i}{\hbar}H_0 t}\psi\rangle = e^{i\frac{\pi}{4}d}\int_{\mathbb{R}^d}\bar{\phi}(e^{i\frac{\pi}{4}}x)\int_{C_t}\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t)+e^{i\frac{\pi}{4}}x)W(d\omega)dx$$

$\square$

By restricting the time interval $[0,t]$, it is possible to generalize the previous result to the case where H_0 is the harmonic oscillator Hamiltonian, given on $\psi \in C_0^2(\mathbb{R}^d)$ by

$$H_0\psi(x) = -\frac{\hbar^2}{2}\Delta\psi(x) + \frac{1}{2}x\Omega^2 x\psi(x), \tag{24}$$

where Ω is a $d \times d$ symmetric positive matrix and Ω_j, $j = 1, \ldots, d$ are its eigenvalues.

Theorem 7. *Let $\bar{\phi}, \psi \in L^2(\mathbb{R}^d)$ satisfying condition 1 of corollary 1. Let us assume that there exists a positive constant $C \in \mathbb{R}^+$ such that $\forall z \in \bar{D}_{0,\frac{\pi}{2}}$ and $\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ the following inequality holds:*

$$|\bar{\phi}(\sqrt{z}\sqrt{\hbar}x)\psi(\sqrt{z}\sqrt{\hbar}(x+y))|e^{\frac{|x|^2}{2}} \leq C.$$

Let us assume moreover that for any $j = 1, \ldots, d$ the time t satisfies the following inequalities:

$$\Omega_j t < \frac{\pi}{2}, \qquad 1 - \Omega_j \tan(\Omega_j t) > 0 \tag{25}$$

Then

$$\langle \phi, e^{-\frac{i}{\hbar}H_0 t}\psi\rangle = e^{i\frac{\pi}{4}d}\hbar^{d/2}\int_{\mathbb{R}^d}\bar{\phi}(e^{i\frac{\pi}{4}}\sqrt{\hbar}x)\int_{C_t}\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t)+e^{i\frac{\pi}{4}}\sqrt{\hbar}x)$$
$$e^{\frac{1}{2}\int_0^t(\omega(s)+x)\Omega^2(\omega(s)+x)ds}W(d\omega)dx \tag{26}$$

where H_0 is the harmonic oscillator Hamiltonian (24).

Proof: By the Feynman-Kac formula and a change of variable, for any $z \in \mathbb{R}^+$ one has

$$\langle \phi, e^{-\frac{z}{\hbar}H_0 t}\psi\rangle = z^{d/2}\hbar^{d/2}\int_{\mathbb{R}^d}\bar{\phi}(\sqrt{z}\sqrt{\hbar}x)\int_{C_t}\psi(\sqrt{\hbar}\sqrt{z}(\omega(t)+x))e^{\frac{|x|^2}{2}}$$
$$e^{-\frac{z^2}{2}\int_0^t(\omega(s)+x)\Omega^2(\omega(s)+x)ds}W(d\omega)e^{-\frac{|x|^2}{2}}dx \tag{27}$$

By the analyticity of the semigroup generated by the harmonic oscillator Hamiltonian, the left hand side is an holomorphic function of $z \in D-\pi/2, \pi/2$ and continuous for $z \in \bar{D}-\pi/2, \pi/2$.
The right hand side satisfies, by the stated assumptions on ϕ, ψ, the following bound for any $z \in \bar{D}_{0,\frac{\pi}{2}}$:

$$
\begin{aligned}
|z^{d/2}\hbar^{d/2} \int_{\mathbb{R}^d} \bar{\phi}(\sqrt{z}\sqrt{\hbar}x) \int_{C_t} \psi(\sqrt{\hbar}\sqrt{z}(\omega(t)+x)) e^{\frac{|x|^2}{2}} \\
e^{-\frac{z^2}{2}\int_0^t (\omega(s)+x)\Omega^2(\omega(s)+x)ds} W(d\omega) e^{-\frac{|x|^2}{2}} dx| \\
\leq |z|^{d/2}|\hbar|^{d/2} C \int_{\mathbb{R}^d \times C_t} e^{\frac{1}{2}\int_0^t (\omega(s)+x)\Omega^2(\omega(s)+x)ds} W(d\omega) e^{-\frac{|x|^2}{2}} dx
\end{aligned}
$$

By the assumption (25) the latter integral is convergent (see [4] and [23] for more details on this estimate).
By Fubini's and Morera's theorems, the right hand side of (27) is an holomorphic function of $z \in D_{0,\pi/2}$ and continuous for $z \in \bar{D}_{0,\pi/2}$, which coincides with the function $z \mapsto \langle \phi, e^{-\frac{z}{\hbar}H_0 t}\psi \rangle$ on $\mathbb{R}^+$. By the uniqueness of analytic continuation one gets for $z = i$ equation (26).

□

We are now going to see to which extent the results of corollary 1 and of theorem 7 can be generalized to the case where the free Hamiltonian resp. the harmonic oscillator Hamiltonian are replaced by the anharmonic oscillator Hamiltonian with polynomial potential (2).
Formally, by substituting $z = i$ also on the right hand side of (20) one obtains the following expression:

$$
\begin{aligned}
e^{i\frac{\pi}{4}d} \int_{\mathbb{R}^d} \bar{\phi}(e^{i\frac{\pi}{4}}x) \int_{C_t} e^{-\frac{e^{i(M+1)\frac{\pi}{2}}\lambda}{\hbar}\int_0^t |\sqrt{\hbar}\omega(s)+x|^{2M} ds} \\
\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t) + e^{i\frac{\pi}{4}}x) W(d\omega) dx := I_t^i(\phi, \psi) \quad (28)
\end{aligned}
$$

Theorem 8. *Let the vectors $\phi, \psi \in L^2(\mathbb{R}^d)$ satisfy the assumptions of corollary 1 and let the degree $2M$ of the polynomial potential V_{2M} be such that $Re(e^{i(M+1)\frac{\pi}{2}}) \geq 0$. Then the integral $I_t^i(\phi, \psi)$ in equation (28) is well defined and satisfies the following inequality*

$$
|I_t^i(\phi, \psi)| \leq \| \phi_i \| \| \psi_i \|,
$$

where $\phi_i(x) := |\bar{\phi}(e^{i\frac{\pi}{4}}x)|, \psi_i(x) := |\psi(e^{i\frac{\pi}{4}}x)|$. Moreover, if $\phi, \psi \in D(H)$ and $H\phi, H\psi$ satisfy the assumptions of corollary 1, the functional $I_t^i(\phi, \psi)$ satisfies the Schrödinger equation (3) in the following weak sense:

$$
I_0^i(\phi, \psi) = \langle \phi, \psi \rangle, \quad (29)
$$

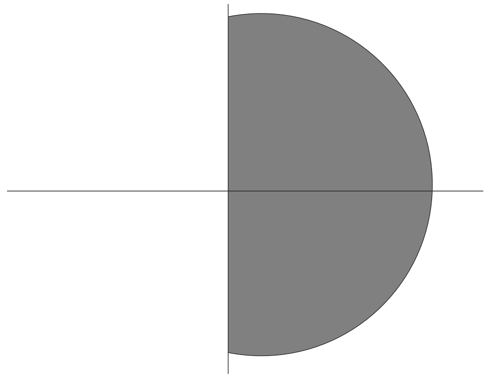

Figure 1: the sector $D_{-\pi/2,\pi/2}$

$$i\hbar\frac{d}{dt}I^i_t(\phi,\psi)=I^i_t(\phi,H\psi)=I^i_t(H\phi,\psi). \tag{30}$$

Proof: The first part of the theorem follows by a direct estimate. Equation (29) is a consequence of corollary 1, while equation 30 follows from Ito's formula. □

A stronger result, namely the equality

$$I^i_t(\phi,\psi)=\langle\phi,e^{-\frac{i}{\hbar}Ht}\psi\rangle \tag{31}$$

cannot in general be proved in the case where H is given by (2). In fact, the analyticity argument used in the proof of corollary 1 cannot in general be applied to the proof of equality (31), as the following considerations show.

By theorem 5 the function $f_1:\bar{D}_{-\pi/2,\pi/2}\to\mathbb{C}$, defined by $f_1(z)=\langle\phi,e^{-\frac{z}{\hbar}H}\psi\rangle$, is analytic on the sector $D_{-\pi/2,\pi/2}$ (shown in figure 1) and continuous on $\bar{D}_{-\pi/2,\pi/2}$ (as H has a positive spectrum and generates an analytic semigroup). On the other hand, as we have already seen, on the positive real line $\mathbb{R}^+$ the function f_1 can be expressed in terms of a functional integral by means of the Feynman-Kac formula:

$$\langle\phi,e^{-\frac{z}{\hbar}H}\psi\rangle=z^{d/2}\int_{\mathbb{R}^d}\bar{\phi}(\sqrt{z}x)\int_{C_t}e^{-\frac{z^{N+1}\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega(s)+x|^{2N}ds}$$
$$\psi(\sqrt{\hbar}\sqrt{z}\omega(t)+\sqrt{z}x)W(d\omega)dxdx \tag{32}$$

and the right hand side of (32) is well defined, if ϕ,ψ are sufficiently regular, when $Re(z^{M+1})\geq 0$, i.e for $z=|z|e^{i\theta}$, with $-\frac{\pi}{2(M+1)}+k\frac{2\pi}{M+1}\leq\theta\leq\frac{\pi}{2(M+1)}+k\frac{2\pi}{M+1}$, $k\in\mathbb{Z}$. i.e. on $M+1$ different sectors of the complex plane. In particular, the right hand side of (32) defines $M+1$ different holomorphic functions g_k with $k=0,\dots,M$,

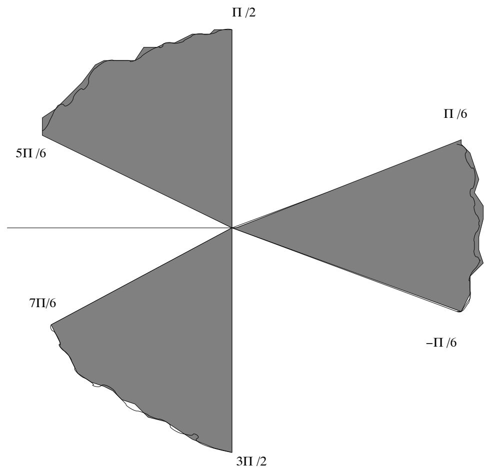

Figure 2: The set of definition of the integral $I(z)$

each of them defined on a different sector of the complex plane, i.e. $D_k := D_{-\frac{\pi}{2(M+1)}+k\frac{2\pi}{M+1}, \frac{\pi}{2(M+1)}+k\frac{2\pi}{M+1}}$, $k = 0, \dots, M$. As the $M+1$ open sectors are disjoint (actually the intersection of their closures contain a unique point), we cannot consider $g_0, \dots, g_M$ as the same analytic function defined on different regions of the complex plane. In particular the analyticity properties of the left and the right hand side of (32) allows to extend the Feynman-Kac formula, if the condition of theorem 6 are satisfied, to all the values of $z \in \bar{D}_{-\frac{\pi}{2(M+1)}, \frac{\pi}{2(M+1)}}$. This sector does not include $z = i$ (unless one considers the trivial case $M = 0$).

For instance, if $2M = 4$ (the quartic oscillator case), the integral on the right hand side of (32) becomes

$$I(z) := z^{d/2} \int_{\mathbb{R}^d} \bar{\phi}(\sqrt{z}x) \int_{C_t} e^{-\frac{z^3\lambda}{\hbar}\int_0^t |\sqrt{\hbar}\omega(s)+x|^{2N} ds} \psi(\sqrt{\hbar}\sqrt{z}\omega(t) + \sqrt{z}x) W(d\omega) dx. \quad (33)$$

Under analyticity and growing conditions on ϕ, ψ, the integral $I(z)$ is well defined on three sectors $\bar{D}_{-\frac{\pi}{6}, \frac{\pi}{6}} \cup \bar{D}_{\frac{\pi}{2}, \frac{5\pi}{6}} \cup \bar{D}_{\frac{7\pi}{6}, \frac{3\pi}{2}}$ of the complex z plane shown in figure 2. In particular the integral $I(z)$ in (33) defines three analytic function g_0, g_1, g_2 defined respectively on the disjoints domains $D_{-\frac{\pi}{6}, \frac{\pi}{6}}$, $D_{\frac{\pi}{2}, \frac{5\pi}{6}}$ and $D_{\frac{7\pi}{6}, \frac{3\pi}{2}}$. They are analytic on their domains of definition and continuous on the boundaries. However the information we have does not allow to prove that g_0, g_1, g_2 are the *same analytic function*, since the intersection of the closure of their definition domains is a single point, i.e. $\bar{D}_{-\frac{\pi}{6}, \frac{\pi}{6}} \cap \bar{D}_{\frac{\pi}{2}, \frac{5\pi}{6}} \cap \bar{D}_{\frac{7\pi}{6}, \frac{3\pi}{2}} = \{0\}$.
We can then only say that equation (32), i.e the equality $f(z) = I(z)$,

holds for z belonging to $\bar{D}_{-\frac{\pi}{6},\frac{\pi}{6}}$, but nothing can be said as it stands for $z = i$. By these considerations, the Feynman path integral representation for the weak solution of the Schrödinger equation studied in [4] has to be interpreted in the weak sense of theorem 8.

The difficulties in the investigation on the relations between the functional integral (28) and the matrix elements of the Schrödinger group $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$ can be better understood by means of the following simplified model.
Let us consider the two functions of the complex variable λ,

$$f_1(\lambda) := \int e^{i\lambda x^4 + ix^2} dx$$

$$f_2(\lambda) := e^{i\pi/4} \int e^{-i\lambda x^4 - x^2} dx$$

defined and analytic respectively on $D_1 = \{Im(\lambda > 0)\}$ and $D_2 = \{Im(\lambda < 0)\}$.

The function f_1, for $\lambda \in \mathbb{R}$ $\lambda < 0$ can be seen as the one dimensional analogue of $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi\rangle$, while function f_2 as the one dimensional analogue of the functional integral (28).

It is possible to verify by means of a rotation of the integration contour in the complex plane, that for $\lambda \in \mathbb{R}^+$ one has $f_1(\lambda) = f_2(\lambda)$. For instance if $\lambda = 1$ one has $\int e^{ix^4+ix^2}dx = e^{i\pi/4}\int e^{-ix^4-x^2}dx$.

The extension of the equality between f_1 and f_2 on the negative real line is, on the other hand, not possible. Indeed it is possible to see that f_1 and f_2 are different branches on an analytic multivalued function defined on a Riemann sheet.

A transformation of variable allows us to represent the two functions in a fashion allowing us to enlighten their analyticity properties and the nature of the singularity in $\lambda = 0$. Indeed when $\lambda \in \mathbb{R}^+$, the following equality holds

$$f_2(\lambda) = e^{i\pi/4} \int e^{-i\lambda x^4 - x^2} dx = \frac{e^{i\pi/4}}{\lambda^{1/4}} \int e^{-ix^4 - \frac{x^2}{\lambda^{1/2}}} dx.$$

The right hand side is an analytic function in the region

$$\left\{\lambda \in \mathbb{C}\,,\, \lambda = |\lambda| e^{i\phi} :\ |\lambda| > 0, -\pi < \phi < \pi\right\}.$$

It is continuous on the boundary of its analyticity domain but it is multivalued and it assumes different values approaching the negative real axis from above and from below:

$$f_2(|\lambda| e^{i\pi}) = \int e^{i\lambda x^4 + ix^2} dx$$

$$f_2(|\lambda|e^{-i\pi}) = e^{i\pi/2}\int e^{i\lambda x^4 - ix^2}dx$$

By a rotation technique it is easy to verify that the latter integral is equal to

$$e^{i\pi/2}\int e^{i\lambda x^4 - ix^2}dx = e^{i\pi/4}\int e^{-i\lambda x^4 - x^2}dx.$$

In other words we can say that the two integrals $\int e^{i\lambda x^4 + ix^2}dx$ and $e^{i\pi/4}\int e^{-i\lambda x^4 - x^2}dx$ do not coincide on the negative real line: they are different branches of the same analytic but multivalued function. In a similar way, the functional integral (28) and the matrix elements of the Schrödinger group can be interpreted, as functions of the complex variable λ, as different branches of the same analytic but multivalued function.

Remark 3. *Despite the problems described so far, in the literature some particular cases have been handled by means of different techniques. In* [22]*, equality* (31) *has been proved in the case where H is the inverse quartic oscillator Hamiltonian*

$$H_0\psi(x) = -\frac{\hbar^2}{2}\Delta\psi(x) + \frac{1}{2}x\Omega^2 x\psi(x) - \lambda|x|^4\psi(x), \qquad \psi \in C_0^2(\mathbb{R}^d), \lambda \in \mathbb{R}^+,$$

by means of an analytic continuation technique in the mass parameter.

In [13] *the pointwise solution of the heat equation* (19) *and its functional integral representation have been considered:*

$$(e^{-\frac{z}{\hbar}Ht}\psi)(x) = \int_{C_t} e^{-\frac{z\lambda}{\hbar}\int_0^t|\sqrt{\hbar z}\omega(s)+x|^{2M}ds}\psi_0(\sqrt{\hbar z}\omega(t)+x)W(d\omega). \quad (34)$$

The right hand side of (34) *evaluated for $z = i$ gives*

$$\int_{C_t} e^{-\frac{i\lambda}{\hbar}\int_0^t|\sqrt{\hbar i}\omega(s)+x|^{2M}ds}\psi_0(\sqrt{\hbar i}\omega(t)+x)W(d\omega). \quad (35)$$

For suitable exponents $2M$, namely for $Re(e^{i(M+1)\frac{\pi}{2}}) < 0$, the integral (35) *is well defined and, as proved in* [13] *by means of a probabilistic argument, it represents the pointwise solution of Schrödinger equation.*

Let us consider now the integral (28) and let us assume that the hypothesis of theorem 8 are satisfied. By considering two suitable sets of vectors in $\mathcal{S}_1, \mathcal{S}_2 \subset L^2(\mathbb{R}^d)$, with $\phi \in \mathcal{S}_2$ and $\phi \in \mathcal{S}_1$, it is possible to interpret the integral $I_t^i(\phi, \psi)$ as the matrix element of an evolution operator in $L^2(\mathbb{R}^d)$.

Let us denote by $\mathcal{S}_1$ the subset of $S(\mathbb{R}^d)$ made of the functions $\phi : \mathbb{R}^d \to \mathbb{C}$ of the form

$$\phi(x) = P(x)e^{-\frac{x^2}{2}(1-i)}, \qquad x \in \mathbb{R}^d, \quad (36)$$

and by $\mathcal{S}_2$ the subset of $S(\mathbb{R}^d)$ made of the functions $\phi : \mathbb{R}^d \to \mathbb{C}$ of the form

$$\phi(x) = Q(x)e^{-\frac{x^2}{2}(1+i)} \tag{37}$$

where P and Q are polynomials with complex coefficients. As the Hermite functions form a complete orthonormal system in $L^2(\mathbb{R}^d)$, it is simple to verify that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are dense in $L^2(\mathbb{R}^d)$. Moreover the functions $\phi \in \mathcal{S}_1$ are such that:

(1) the function $z \mapsto \phi(zx)$, $x \in \mathbb{R}^d$, $z \in \bar{D}_{0,\pi/4}$ is analytic on $D_{0,\pi/4}$ and continuous on $\bar{D}_{0,\pi/4}$,
(2) the function $x \mapsto \phi(e^{i\frac{\pi}{4}}x)$, $x \in \mathbb{R}^d$ is in L^2,

while the functions $\phi \in \mathcal{S}_2$ are such that:

(1) the function $z \mapsto \phi(zx)$, $x \in \mathbb{R}^d$, $z \in \bar{D}_{-\pi/4,0}$ is analytic on $D_{-\pi/4,0}$ and continuous on $\bar{D}_{-\pi/4,0}$,
(2) the function $x \mapsto \phi(e^{-i\frac{\pi}{4}}x)$, $x \in \mathbb{R}^d$ belongs to $L^2(\mathbb{R}^d)$.

Let us denote by $T : \mathcal{S}_1 \to \mathcal{S}_2$ the linear operator defined by

$$T\phi(x) = e^{i\frac{\pi}{8}d}\phi(e^{i\frac{\pi}{4}}x), \qquad \phi \in \mathcal{S}_1,$$

and by $T^{-1} : \mathcal{S}_2 \to \mathcal{S}_1$ its inverse, defined by

$$T^{-1}\phi(x) = e^{-i\frac{\pi}{8}d}\phi(e^{-i\frac{\pi}{4}}x), \qquad \phi \in \mathcal{S}_2.$$

By considering two vectors $\phi, \psi \in S_1$ one can easily verify that

$$\langle \phi, T\psi \rangle = \langle T\phi, \psi \rangle. \tag{38}$$

Analogously, by considering two vectors $\phi, \psi \in S_2$, one can easily verify that

$$\langle \phi, T^{-1}\psi \rangle = \langle T^{-1}\phi, \psi \rangle. \tag{39}$$

This implies that, for $\phi \in S_2, \psi \in S_1$, one has $\langle T^{-1}\phi, T\psi \rangle = \langle \phi, \psi \rangle$.

Let $H_T : \mathcal{S}_2 \to \mathcal{S}_2$ be the operator defined by $-iH_T := THT^{-1}$. It is easy to verify that

$$H_T\phi(x) = -\frac{\hbar^2}{2}\Delta\phi(x) + \lambda e^{i\frac{M+1}{2}\pi}|x|^{2M}\phi(x), \qquad \phi \in \mathcal{S}_2.$$

Theorem 9. *Let $\psi \in \mathcal{S}_1$ and $\phi \in \mathcal{S}_2$. Let us assume that $Re(e^{i\frac{M+1}{2}\pi}) \geq 0$. Then the operator H_T is the restriction to $\mathcal{S}_2$ of the generator A of a strongly continuous contraction semigroup $V(t) = e^{-\frac{1}{\hbar}At}$ and the integral $I_t^i(\phi, \psi)$ given by (28) is equal to the matrix element $\langle T^{-1}\phi, V(t)T\psi \rangle$.*

Proof: Let $V(t)_{t\geq 0}$ be the C_0-contraction semigroup defined by the Feynman-Kac type formula:

$$V(t)\psi(x) := \int_{C_t} e^{-e^{i\frac{M+1}{2}\pi}\frac{\lambda}{\hbar}\int_0^t |\sqrt{\hbar}\omega(s)+x|^{2M}ds}\psi(\sqrt{\hbar}\omega(t) + x)W(d\omega). \tag{40}$$

The operator-theoretic results for semigroups of the form (40) have been investigated in [24, 20] (see also [18], chapter 13.5). In particular the generator A of the semigroup $V(t) = e^{-\frac{t}{\hbar}A}$ is given on smooth vectors $\psi \in S(\mathbb{R}^d)$ by

$$A\psi(x) = -\frac{\hbar^2}{2}\Delta\psi(x) + Q_{2M}(x)\psi(x), \quad Q_{2M}(x) := \lambda e^{i\frac{M+1}{2}\pi}|x|^{2M} \quad (41)$$

with domain

$$D(A) = \{\psi \in H^1(\mathbb{R}^d) \,:\, -\frac{\hbar^2}{2}\Delta\psi + Q_{2M}\psi \in L^2(\mathbb{R}^d)\}. \quad (42)$$

By a direct computation and by taking $\psi \in \mathcal{S}_1$ and $\phi \in \mathcal{S}_2$, one can easily verify that

$$e^{i\frac{\pi}{4}d}\int_{\mathbb{R}^d}\bar{\phi}(e^{i\frac{\pi}{4}}x)\int_{C_t} e^{-\frac{e^{i(M+1)\frac{\pi}{2}}\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega(s)+x|^{2M}ds}$$
$$\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t)+e^{i\frac{\pi}{4}}x)W(d\omega)dx = \langle T^{-1}\phi, V(t)T\psi\rangle$$

so that $I^i_t(\phi,\psi) = \langle T^{-1}\phi, V(t)T\psi\rangle$. □

Remark 4. *Under the assumptions of theorem 9, it is possible to give an alternative proof of theorem 8, i.e. that the integral $I^i_t(\phi,\psi)$ is a weak solution of the Schrödinger equation in the sense that equations* (29) *and* (30) *are satisfied.*
Indeed equation (29) *follows by writing $I^i_0(\phi,\psi)$ as $\langle T^{-1}\phi, T\psi\rangle$ and by equations* (38) *and* (39)*. Equation* (30) *follows from the equality $I^i_t(\phi,\psi) := \langle T^{-1}\phi, e^{-\frac{t}{\hbar}A}T\psi\rangle$. Indeed*

$$i\hbar\frac{d}{dt}\langle T^{-1}\phi, e^{-\frac{t}{\hbar}A}T\psi\rangle = \langle T^{-1}\phi, e^{-\frac{t}{\hbar}A}(-iH_T)T\psi\rangle = \langle T^{-1}\phi, e^{-\frac{t}{\hbar}H_T}TH\psi\rangle.$$

Remark 5. *To prove that $I^i_t(\phi,\psi) = \langle\phi, e^{-\frac{it}{\hbar}H}\psi\rangle$ it would be sufficient to have that $|I^i_t(\phi,\psi)| \le C\|\phi\|\|\psi\|$, or, in other words, that given $\psi \in \mathcal{S}_1$, one has that the vector $e^{-\frac{t}{\hbar}H_T}T\psi$ belongs to the domain of $(T^{-1})^*$, so that*

$$\langle T^{-1}\phi, e^{-\frac{t}{\hbar}H_T}T\psi\rangle = \langle\phi, (T^{-1})^* e^{-\frac{t}{\hbar}H_T}T\psi\rangle.$$

If the inequality $|I^i_t(\phi,\psi)| \le C\|\phi\|\|\psi\|$ holds true, then this would namely imply that there exists a bounded operator $B(t) : L^2 \to L^2$ such that $I^i_t(\phi,\psi) = \langle\phi, B(t)\psi\rangle$ and $B(0) = I$. $I^i_t(\cdot,\cdot)$ defined on $\mathcal{S}_2\times\mathcal{S}_1$ can be extended to $L^2\times L^2$. In particular then $I^i_t(U(t)\phi,\psi)$ makes sense and by differentiating with respect to the time variable t we obtain

$$i\hbar\frac{d}{dt}I^i_t(U(t)\phi,\psi) = I^i_t(U(t)\phi, H\psi) - I^i_t(HU(t)\phi,\psi) = 0$$

so that $I^i_t(U(t)\phi,\psi)=I^i_0(U(0)\phi,\psi)=\langle\phi,\psi\rangle$ *for any* t *and this implies that* $B(t)=U(t)$.

5. The functional integral as asymptotic solution

The present section is devoted to the proof that the functional integral (28) coincides asymptotically both as $t\to 0$ and as $\lambda\to 0$ with the matrix element $\langle\phi, e^{\frac{i}{\hbar}Ht}\psi\rangle$ of the Schrödinger group.

Theorem 10. *Let* $\phi,\psi\in L^2(\mathbb{R}^d)$ *and* $M\in\mathbb{N}$ *satisfy the assumptions of theorem 9. Then as* $t\to 0$ *the integral* $I^i_t(\phi,\psi)$ *and the matrix element* $\langle\phi, e^{\frac{i}{\hbar}Ht}\psi\rangle$ *of the Schrödinger group admit the following asymptotic expansions*

$$I^i_t(\phi,\psi)=\sum a_nt^n,\qquad \langle\phi, e^{\frac{i}{\hbar}Ht}\psi\rangle=\sum b_nt^n,$$

and they coincides, i.e. $a_n=b_n$ $\forall n\in\mathbb{N}$.

Proof: By theorem 9, the functional integral $I^i_t(\phi,\psi)$ can be written as $\langle T^{-1}\phi, e^{-\frac{t}{\hbar}A}T\psi\rangle$, where A is the operator defined by (41) and (42). Under the stated assumptions, the vector $T\psi$ belongs to the domain of A^n $\forall n\in\mathbb{N}$, and $A^nT\psi$ belongs to $\mathcal{S}_2\subset D(T^{-1})$ $\forall n\in\mathbb{N}$. In particular, for any $N\in\mathbb{N}$ one has

$$\langle T^{-1}\phi, e^{-\frac{t}{\hbar}A}T\psi\rangle=\sum_{n=0}^N\frac{1}{n!}\Big(-\frac{t}{\hbar}\Big)^n\langle T^{-1}\phi, A^nT\psi\rangle+R_N,$$

$$R_N=\frac{1}{N-1!}\Big(-\frac{t}{\hbar}\Big)^N\int_0^1u^{N-1}\langle T^{-1}\phi, A^Ne^{-\frac{t}{\hbar}A(1-u)}T\psi\rangle du.$$

The remainder R_N can easily be estimated by means of Schwarz inequality and one has:

$$|R_N|\leq\frac{|t|^N|\hbar|^{-N}}{N!}\|T^{-1}\phi\|\|A^NT\psi\|=O(|t|^N).$$

As $T\psi\in\mathcal{S}_2$, one has $A^nT\psi=(H_T)^nT\psi=i^nTH^n\psi$, so that by equation (39)

$$\langle T^{-1}\phi, e^{-\frac{t}{\hbar}A}T\psi\rangle=\sum_{n=0}^N\frac{1}{n!}\Big(-\frac{it}{\hbar}\Big)^n\langle\phi, H^n\psi\rangle+O(t^N).$$

Analogously

$$\langle\phi, e^{-\frac{it}{\hbar}H}\psi\rangle=\sum_{n=0}^N\frac{1}{n!}\Big(-\frac{it}{\hbar}\Big)^n\langle\phi, H^n\psi\rangle+R'_N,$$

$$R'_N = \frac{1}{N-1!}\Big(-\frac{it}{\hbar}\Big)^N \int_0^1 u^{N-1}\langle \phi, H^N e^{-\frac{it}{\hbar}H(1-u)}\psi\rangle du,$$

with $R'_N = O(t^N)$, and one can easily verify that the asymptotic expansion in powers of t of $I^i_t(\phi,\psi)$ and of $\langle \phi, e^{-\frac{it}{\hbar}H}\psi\rangle$ coincide. □

Remark 6. *The power series of the variable t are not convergent, but only asymptotic. This fact implies that the result theorem 10 is not sufficient to deduce the equality between $I^i_t(\phi,\psi)$ and $\langle\phi, e^{\frac{i}{\hbar}Ht}\psi\rangle$.*

Let us consider now couples of vectors ϕ,ψ satisfying the assumptions of theorem 1 (so that the result of theorem 4 holds) and such that the functional integral (28) is well defined. In fact, it is always possible to find a dense set of vectors in $L^2(\mathbb{R}^d)$ satisfying both conditions. For instance, if $M=2$, it is sufficient to take $\psi\in\mathcal{S}_1$ and $\phi\in\mathcal{S}_2$, while if $M\geq 3$ the fulfillment of hypothesis of theorem 1 implies that the integral (28) is well defined. Under these conditions, it is possible to interpret the functional integral (28) as an asymptotic weak solution of the Schrödinger equation, in the sense of the following theorem.

Theorem 11. *Under the assumptions above, the asymptotic expansion in powers of the coupling constant λ as $\lambda\to 0$ of the functional integral representation* (28) *coincides with the corresponding asymptotic expansion* (14) *of the matrix elements of the Schrödinger group. Moreover the latter is Borel summable.*

Proof: By expanding the functional integral (28) in powers of λ one has

$$I^i_t(\phi,\psi) = \sum_n^{N-1} a_n\lambda^n + R_N,$$

with

$$a_n = \frac{1}{n!}e^{i\frac{\pi}{4}d}\int_{\mathbb{R}^d}\bar\phi(e^{i\frac{\pi}{4}}x)\int_{C_t}\Big(-\frac{e^{i(M+1)\frac{\pi}{2}}\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega(s)+x|^{2M}ds\Big)^n$$
$$\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t)+e^{i\frac{\pi}{4}}x)W(d\omega)dx.$$

and

$$R_N = \frac{\lambda^N}{(N-1)!} e^{i\frac{\pi}{4}d} \int_0^1 (1-u)^{N-1} \int_{\mathbb{R}^d} \bar{\phi}(e^{i\frac{\pi}{4}}x)$$
$$\int_{C_t} \Big(- \frac{e^{i(M+1)\frac{\pi}{2}}\lambda}{\hbar} \int_0^t |\sqrt{\hbar}\omega(s) + x|^{2M} ds \Big)^N e^{-u\frac{e^{i(M+1)\frac{\pi}{2}}\lambda}{\hbar}\int_0^t |\sqrt{\hbar}\omega(s)+x|^{2M} ds}$$
$$\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t) + e^{i\frac{\pi}{4}}x) W(d\omega) dx du.$$

It is easy to verify that $R_N = O(\lambda^N)$. Moreover, by exploiting the symmetry of the integrand, the coefficients a_n can be written as:

$$a_n = e^{i\frac{\pi}{4}d} \Big(- \frac{e^{i(M+1)\frac{\pi}{2}}\lambda}{\hbar} \Big)^n \int_0^t \dots \int_0^t ds_1 \dots ds_n \int_{\mathbb{R}^d} \bar{\phi}(e^{i\frac{\pi}{4}}x)$$
$$\int_{C_t} |\sqrt{\hbar}\omega(s_1) + x|^{2M} ds \dots |\sqrt{\hbar}\omega(s_n) + x|^{2M} ds$$
$$\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t) + e^{i\frac{\pi}{4}}x) W(d\omega) dx.$$

By the result of corollary 1, the latter coincides with the coefficient a_n of the asymptotic expansion (14). On the other hand, by theorem 4, the matrix elements of the Schrödinger group can be obtained in terms of the Borel sum of the asymptotic expansion $\sum a_n \lambda^n$. □

Remark 7. *By a direct computation it is possible to verify that in the case $M = 2$ (the quartic oscillator case) the functional integral $I_t^i(\phi, \psi)$ can be extended to an analytic function of the variable λ in the sector $D_{0,\pi}$ of the complex plane and satisfies there an estimate of the following form*

$$|I_t^i(\phi, \psi) - \sum_n^{N-1} a_n \lambda^n| \leq AC^N |\lambda|^N N!.$$

By Watson-Nevanlinna's theorem, it is possible to recover $I_t^i(\phi, \psi)$ in terms of the coefficients a_n in the asymptotic expansion. This result, combined with the analogous result for $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi \rangle$, is not sufficient to deduce the equality $I_t^i(\phi, \psi) = \langle \phi, e^{-\frac{i}{\hbar}Ht}\psi \rangle$, as the two function are defined as the Borel sums of the same asymptotic expansion but on different regions of the complex plane (the left hand side on $D_{0,\pi}$ and the right hand side on $D_{\pi,o}$). Indeed, let us consider two functions $f_1(z)$ and $f_2(z)$ of the complex variable z, defined and holomorphic resp. in $D_{0,\pi}$ and $D_{\pi,0}$, admitting as $z \to 0$ the same asymptotic expansion and *estimate uniformly in their analyticity domains:*

$$f_1(z) \sim \sum a_n z^n, \qquad |f_1(z) - \sum_n^{N-1} a_n z^n| \leq A_1 C_1^N |z|^N N!,$$

$$f_2(z) \sim \sum a_n z^n, \qquad |f_2(z) - \sum_n^{N-1} a_n z^n| \leq A_2 C_2^N |z|^N N!.$$

The function f_1, f_2 can be recovered by their asymptotic expansion $\sum a_n z^n$ by means of the following procedure. Let us define two functions $g_1(z)$ and $g_2(z)$ of the complex variable $z \in D_{-\pi/2,\pi/2}$ defined by

$$g_1(z) := f_1(iz), \qquad g_2(z) := f_2(-iz), \qquad z \in D_{-\pi/2,\pi/2}.$$

g_1 and g_2 admit the following asymptotic expansion and estimate:

$$g_1(z) \sim \sum i^n a_n z^n, \qquad |g_1(z) - \sum_n^{N-1} i^n a_n z^n| \leq A_1 C_1^N |z|^N N!,$$

$$g_2(z) \sim \sum (-i)^n a_n z^n, \qquad |g_2(z) - \sum_n^{N-1} (-i)^n a_n z^n| \leq A_2 C_2^N |z|^N N!.$$

By theorem 3 they are both Borel summable, i.e. formally:

$$g_1(z) = \frac{1}{z} \int_0^\infty e^{-t/z} \sum \frac{i^n a_n}{n!} t^n dt, \tag{43}$$

$$g_2(z) = \frac{1}{z} \int_0^\infty e^{-t/z} \sum \frac{(-i)^n a_n}{n!} t^n dt. \tag{44}$$

One would have $f_1(z) = f_2(z)$ for $z \in \mathbb{R}^+$, if $g_1(i\rho) = g_2(-i\rho)$ for $\rho \in \mathbb{R}^+$, however the Borel summability of the asymptotic expansion $\sum a_n z^n$, in particular the relations (43) *and* (44)*, are by themselves not sufficient to deduce this equality.*

Acknowledgments

The hospitality of the Mathematics Institutes of Trento and of Bonn Universities is gratefully acknowledged, as well as the financial support of the F. Severi fellowship of I.N.d.A.M.

References

[1] S. Albeverio, R. Høegh-Krohn, S. Mazzucchi Mathematical theory of Feynman path integrals. An Introduction. 2nd and enlarged edition. Lecture Notes in Mathematics, Vol. 523. Springer-Verlag, Berlin-New York (2008).

[2] S. Albeverio, S. Mazzucchi, Generalized Fresnel Integrals, Bull. Sci. Math. 129, no. 1, 1–23 (2005).

[3] S. Albeverio, S. Mazzucchi, Generalized infinite-dimensional Fresnel Integrals, C. R. Acad. Sci. Paris 338 n.3, 255–259 (2004).

[4] S. Albeverio, S. Mazzucchi, Feynman path integrals for polynomially growing potentials, J. Funct. Anal. 221 no.1, 83–121 (2005).

[5] S. Albeverio, S. Mazzucchi, Some New Developments in the Theory of Path Integrals, with Applications to Quantum Theory, J. Stat. Phys. 115 n.112, 191–215 (2004).

[6] S. Albeverio, S. Mazzucchi, Feynman Path integrals for time-dependent potentials, in : "Stochastic Partial Differential Equations and Applications -VII", G. Da Prato and L. Tubaro eds, Lecture Notes in Pure and Applied Mathematics, vol. 245, Taylor & Francis, (2005), pp 7-20.

[7] S. Albeverio, S. Mazzucchi, Feynman path integrals for the time dependent quartic oscillator. C. R. Acad. Sci. Paris 341, no. 10, 647–650. (2005).

[8] S. Albeverio, S. Mazzucchi, The time dependent quartic oscillator - a Feynman path integral approach. J. Funct. Anal. 238 , no. 2, 471–488 (2006).

[9] S. Albeverio, S. Mazzucchi, The trace formula for the heat semigroup with polynomial potential, SFB-611-Preprint no. 332, Bonn (2007).

[10] C. Bender, T. Wu, Anharmonic oscillator. Phys. Rev. (2) 184 (1969) 1231–1260.

[11] I. M. Davies, A. Truman, On the Laplace asymptotic expansion of conditional Wiener integrals and the Bender-Wu formula for x^{2N}-anharmonic oscillators. J. Math. Phys. 24 , no. 2, 255–266 (1983).

[12] R.H. Cameron, A family of integrals serving to connect the Wiener and Feynman integrals, J. Math. and Phys. 39, 126–140 (1960).

[13] H. Doss, Sur une Résolution Stochastique de l'Equation de Schrödinger à Coefficients Analytiques. *Commun. Math. Phys.*, 73, 247-264 (1980).

[14] K.-J. Engel, R. Nagel, One-parameter semigroups for linear evolution equations. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000.

[15] Gross L., Abstract Wiener Spaces, Proc. 5^{th} Berkeley Symp. Math. Stat. Prob. 2, (1965) 31-42.

[16] G. H. Hardy, Divergent series, Oxford University Press, London (1963).

[17] T. Hida, H.H. Kuo, J. Potthoff, L. Streit, *White Noise* Kluwer, Dordrecht (1995).

[18] G.W. Johnson and M.L. Lapidus, *The Feynman integral and Feynman's operational calculus.* Oxford University Press, New York (2000).

[19] G. Kallianpur, D. Kannan, R.L. Karandikar *Analytic and sequential Feynman integrals on abstract Wiener and Hilbert spaces, and a Cameron Martin Formula*, Ann. Inst. H. Poincaré, Prob. Th. **21** (1985), 323-361.

[20] T. Kato, On some Schrödinger operators with a singular complex potential. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 5 , no. 1, 105–114 (1978).

[21] Kuo H.H., Gaussian Measures in Banach Spaces, Lecture Notes in Math., Springer-Verlag Berlin-Heidelberg-New York, (1975).

[22] S. Mazzucchi, Feynman path integrals for the inverse quartic oscillator, J. Math. Phys. 49, 9 (2008), 093502 (15 pages)

[23] S. Mazzucchi, Mathematical Feynman Path Integrals and Applications. World Scientific Publishing, Singapore (2009)

[24] E. Nelson, *Feynman integrals and the Schrödinger equation*, J. Math. Phys. **5** (1964), 332-343.

[25] F. Nevanlinna. *Zur Theorie der asymptotischen Potenzreihen. Ann. Acad. Sci. Fenn. (A)*, 12 (3) (1919), 1-81.

[26] M. Reed, B. Simon, Methods of modern mathematical physics. II. Fourier analysis, self-adjointness. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[27] M. Reed, B. Simon, Methods of modern mathematical physics. IV. Analysis of operators. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[28] B. Simon, Coupling Constant Analiticity for the Anharmonic Oscillator. Annals of Physics 58, 76-136 (1970).

[29] B. Simon, Functional integration and quantum physics. Second edition. AMS Chelsea Publishing, Providence, RI, 2005.

[30] A. Sokal. An improvement of Watson's theorem on Borel summability. *J. Math. Phys.* **21**, 261-263, 1980.

[31] K. Yajima, Smoothness and Non-Smoothness of the Fundamental Solution of Time Dependent Schrödiger Equations, Commun. Math. Phys. 181, 605–629 (1996).

* Institut für Angewandte Mathematik, Wegelerstr. 6, 53115 Bonn, HCM; SFB611, BIBOS; IZKS; Dipartimento di Matematica, Università di Trento, 38050 Povo, Italia

** Cerfim (Locarno), Acc. Arch.(USI) (Mendrisio).

*** F. Severi fellow.